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Fractally shaped acceptance domains of quasiperiodic square–triangle tilings with dodecagonal symmetry

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We generate a quasiperiodic, dodecagonally symmetric tiling of the plane by squares and equilateral triangles embedded in a higher-dimensional periodic structure. Starting from a 4D lattice frequently used for the embedding of dodecagonal structures, we iteratively construct an acceptance domain (AD) for a quasiperiodic dodecagonal point set which proves to be the vertex set of a square–triangle tiling. It turns out that our procedure leads to fractally bounded ADs but leaves enough freedom to generate several different local isomorphism classes.

1. Introduction

Quasicrystals with dodecagonal symmetry have been discovered almost contemporaneously [1] with the well-known icosahedral alloys [2]. As indicated by high resolution electron micrographs (HREM) [1,3], these structures closely resemble decorations of periodically stacked dodecagonal plane tilings containing squares and equilateral triangles accompanied by a certain amount of 30° rhombi. In analogy to the celebrated decagonal Penrose pattern [4,5], there have been various proposals to construct quasiperiodic dodecagonal tilings with these constituents [6,7]. Since the amount of rhombi seen in the HREM [3] is really small and could probably be considered as a defect structure in an ideal square–triangle tiling, one has tried to throw out the rhombi completely from the tilings. Especially in ref. [6] one finds such a tiling constructed by deflation but involving random choices at each recursion step.

In this paper, we present a completely *deterministic* ansatz for the generation of dodecagonal square–triangle tilings by the well-known cut-and-project method from a lattice in 4D space. This construction automatically guarantees the *quasiperiodicity* of the tiling and can easily be implemented in a computer code for the pattern generation.

2. The pattern and its vertex acceptance domain

The general framework in which our construction takes place is standard (see, e.g., ref. [8]); we outline it here because we have to introduce some notation anyhow. We use the following 4D lattice.

Let $\mathbb{E}_\parallel$ be the 2D subspace of the physical space which carries the quasiperiodic part of the quasicrystalline structure. It can be embedded in a 4D space $\mathbb{E}_\parallel \oplus \mathbb{E}_\perp$ where $\mathbb{E}_\perp$ denotes the 2D complementary "internal" space. The 4D lattice Λ is defined to be the integral linear span of the vectors

$$\begin{aligned} &e_j = e_j^\parallel + e_j^\perp \qquad (j = 1, \ldots, 12)\,, \\ &\quad \text{with } e_j^\parallel = (D_\parallel^{(12)})^{j-1} e_1^\parallel\,, \quad e_j^\perp = (D_\perp^{(12)})^{5(j-1)} e_1^\perp\,, \end{aligned} \tag{1}$$

where $e_1^\parallel$, $e_1^\perp$ are some chosen unit vectors in $\mathbb{E}_\parallel$, $\mathbb{E}_\perp$, respectively, and $D_\parallel^{(12)}$, $D_\perp^{(12)}$ denote the rotation by $2\pi/12$ in $\mathbb{E}_\parallel$, $\mathbb{E}_\perp$, respectively. We denote a general lattice point by q and its projection into $\mathbb{E}_\parallel$, $\mathbb{E}_\perp$ by $q_\parallel$, $q_\perp$, respectively.

A discrete set of points in $\mathbb{E}_\parallel$ can then be obtained as follows. Let V be a nonempty bounded subset of $\mathbb{E}_\perp$ which is the closure of its interior ($\overline{V^0} = V$). For every $c_\perp \in \mathbb{E}_\perp$ define

$$\mathcal{P}_{c_\perp}^V = \{ q_\parallel \mid c_\perp \in q_\perp + V\,, \quad q \in \Lambda \}\,. \tag{2}$$

(This is well defined as one checks that Λ projects one-to-one into $\mathbb{E}_\parallel$ and $\mathbb{E}_\perp$.) The set V serves as the acceptance domain (AD) for the points of Λ to be projected into $\mathbb{E}_\parallel$ to become members of $\mathcal{P}_{c_\perp}^V$. Such a point set is *quasiperiodic* (the Fourier transform of δ-scatterers placed on the set $\mathcal{P}_{c_\perp}^V$ consists of Bragg peaks on a finitely generated module) if $c_\perp$ is not contained in a boundary of some $q_\perp + V$. Furthermore, if V is invariant under the dodecagonal rotation $D_\perp^{(12)}$, $\mathcal{P}_{c_\perp}^V$ has generalized dodecagonal symmetry (every finite configuration of $\mathcal{P}_{c_\perp}^V$ occurs in every orientation reachable by $D_\parallel^{(12)}$; diffraction intensities of symmetry preserving decorations of $\mathcal{P}_{c_\perp}^V$ are dodecagonally symmetric). Our task now is to define the AD V in a suitable way such that the sets $\mathcal{P}_{c_\perp}^V$ are the vertex sets of square–triangle tilings.

Consider the following system of affine similarity transformations in $\mathbb{E}_\perp$:

$$\begin{aligned} h_j(x) &:= e_1^\perp + (2-\sqrt{3})(D_\perp^{(12)})^{2(j-1)} x \qquad (j = 1, \ldots, 6)\,, \\ h_7(x) &:= e_4^\perp - e_9^\perp + (2-\sqrt{3}) x\,, \\ h_8(x) &:= e_4^\perp - e_9^\perp + (2-\sqrt{3})(D_\perp^{(12)})^{11} x\,, \\ h_9(x) &:= e_{10}^\perp - e_5^\perp + (2-\sqrt{3}) x\,, \\ h_{10}(x) &:= e_{10}^\perp - e_5^\perp + (2-\sqrt{3}) D_\perp^{(12)} x\,. \end{aligned} \tag{3}$$

We then define

$$K_0 := \text{conv}\{0, (\sqrt{3}-1)e_1^\perp, e_4^\perp - e_9^\perp, e_{10}^\perp - e_5^\perp\} \quad \text{(convex closure)} \tag{4}$$

and

$$K_{n+1} := K_n \cup \bigcup_{j=1}^{10} h_j(K_n)\,, \quad K := \overline{\bigcup_{n=0}^{\infty} K_n}\,, \quad V := \bigcup_{j=1}^{12} (D_\perp^{(12)})^j(K)\,. \tag{5}$$

Obviously, V is bounded and invariant under $D_\perp^{(12)}$, and it can easily be seen that also $\overline{V^0} = V$, therefore, the basic demands are fulfilled. More tedious, but possible, is the proof of the following facts: (i) V and $q_\perp + V^0$ overlap only if $|q_\parallel| \geq 1$; (ii) V is covered by the images of $\triangle := V \cap (e_1^\perp + V) \cap (e_3^\perp + V)$ under the rotations $(D_\perp^{(12)})^j$, $j = 1, \ldots, 12$; (iii) $\triangle$ is covered by $(D_\perp^{(12)})^{10}(\triangle)$ and $\square := V \cap (e_3^\perp + V) \cap (e_6^\perp + V) \cap (e_3^\perp + e_6^\perp + V)$; (iv) $\square$ is covered by $\triangle$ and $(D_\perp^{(12)})^9(\square)$.

From this and the definition of V and $\mathcal{P}_{c_\perp}^V$, taking into account dodecagonal symmetry, one concludes that (i) the minimal distance of points in $\mathcal{P}_{c_\perp}^V$ is at least 1; (ii) every point of $\mathcal{P}_{c_\perp}^V$ is part of a configuration of three points in $P_{c_\perp}^V$ spanning an equilateral triangle of bond length 1; (iii) each bond of such a triangle is also a bond of either a second triangular configuration in $P_{c_\perp}^V$ or a configuration of four points in $\mathcal{P}_{c_\perp}^V$ spanning a square of bond length 1; (iv) each bond of such a square is also a bond of either a second square or a

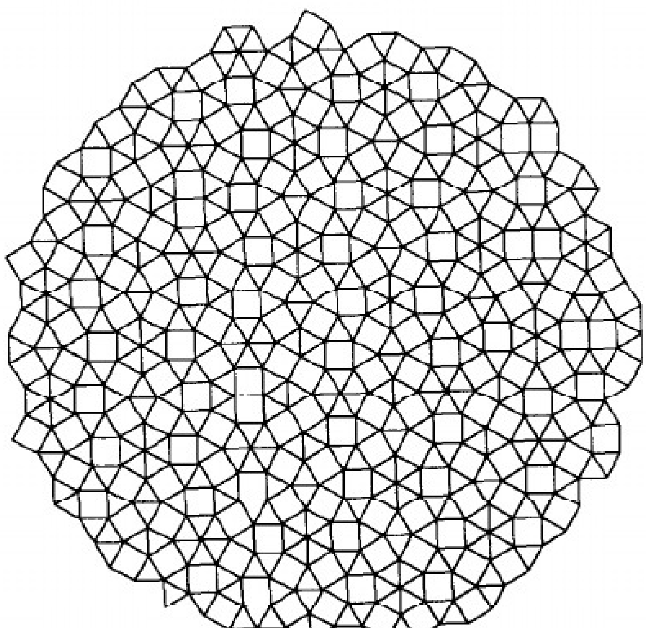

Fig. 1. Typical finite portion of a quasiperiodic dodecagonal square–triangle tiling.

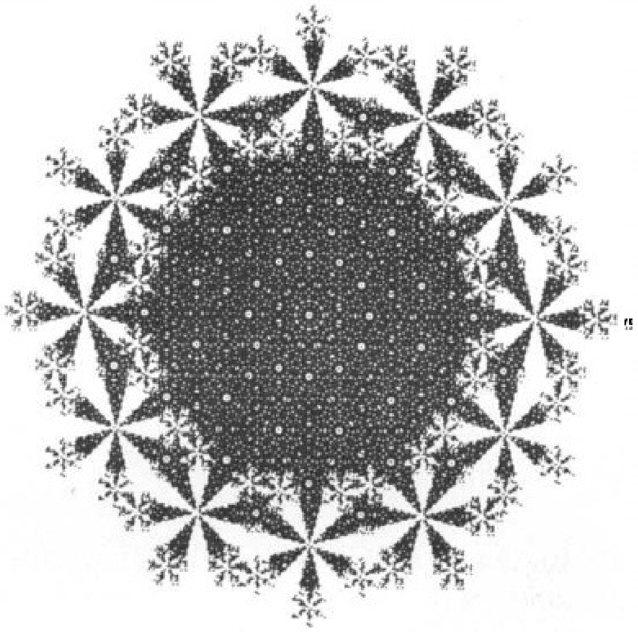

Fig. 2. The acceptance domain filled by the projection of the lift of 32000 vertices in perpendicular space. The circle shaped holes in the bulk are finite size effects.

triangle. Collecting these properties one easily argues that indeed $\mathcal{P}^{V}_{c_\perp}$ is the vertex set of a square–triangle tiling, thereby achieving our goal.

As the very definition of the pattern is constructive, we have no difficulty to visualize finite portions of it (fig. 1). In order to demonstrate the AD V and the fractal nature of its boundary, we chose a method suitable also for the determination of presumably existing ADs of given square–triangle patterns (e.g., extracted from a HREM, if large enough). For a tiling by squares and equilateral triangles it is always possible to consider its vertex set as the projection image of a uniquely determined subset of Λ. One is free to project this subset into $\mathbb{E}_\perp$, and if the vertex set is a set $\mathcal{P}^{V}_{c_\perp}$, $-V$ is densely and uniformly filled by this projection. In fig. 2 we have depicted the results of this procedure applied to a finite portion of 32000 vertices of our pattern. One observes the dodecagonal symmetry of V and the "windmill" structure on every scale $(2-\sqrt{3})^n$ in its boundary region. (The circle shaped holes in the bulk manifest finite size effects resembling the deterministic rather than random nature of this procedure and are not present in V.)

3. Comments

The acceptance domain we used for the generation of our pattern is rather complicated and one may ask if one could achieve the same goal using a polyhedral AD as well. But a deeper look into the circumstances leading to the definition of V seem to indicate that this is impossible for a dodecagonal square–triangle tiling at least if its vertex set is obtained by only one type of AD. On the other hand, there is some freedom in the choice of the starting set K_0 and the system of transformations h_j; this results in the existence of a *continuous variety* of quasiperiodic square–triangle tilings with dodecagonal symmetry. In fact, it is possible to fractalize the boundary of V (which contains straight line segments in the version depicted in fig. 2) completely thereby obtaining a dodecagonal square–triangle tiling with very simple inflation/deflation symmetries, which will be presented in a forthcoming publication.

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Note added in proof

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