SUBSTITUTIONAL TILINGS WITH NON-EMPTY LIMIT TRANSLATIONAL MODULES, BUT WITH VANISHING BRAGG INTENSITIES

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ABSTRACT

Two 1D sequences are discussed, which prove to have vanishing BRAGG intensities with respect to $k = 0$, olthough their limit translational modules are not empty. A 3D embedding is given as well, in order to understand this phenomenon from the cut-and-project scenario. Both sequences can be interpreted as cross-sections of 2D 7-fold resp. 14-fold tilings. One of these possesses local perfect matching rules.

1. Introduction

Recently, a 14-fold tiling with perfect matching rules was discussed[1]. It seemed to be somehow accademical, for its scaling factor is no PISOT-VIJAYARAGHAVAN (PV) number (algebraic conjugates of modulus ≥ 1 do exist). But a deeper insight into this example results in rather unexpected findings, given below. For simplicity, let us start in 1D. To show various aspects we handle two different examples at the same time. A broader outline will be published elsewhere[2].

2. Two 1D Substitutions

First, we define $\vartheta = 1 + 2\cos(2\pi/7)$, $\eta = 1 + 2\cos(\pi/7)$, and $\lambda = \eta - 1$, the maximal roots of the minimal polynomials $m_\vartheta = x^3 - 2x^2 - x + 1$, $m_\eta = x^3 - 4x^2 + 3x + 1$, and $m_\lambda = x^3 - x^2 - 2x + 1$. Since $\eta = \vartheta^2 - \vartheta$ and $\vartheta = \eta^2 - 2\eta$ it is obvious that $\mathbb{Z}[\vartheta] = \mathbb{Z}[\eta] = \mathbb{Z}[\lambda]$. Only ϑ is a PV number. Now, we consider the substitutions $\sigma_1 : L \to LM\overline{L}$, $M \to L\overline{M}S$, $S \to \overline{M}S$, $\overline{L} \to LM\overline{L}$, $\overline{M} \to \overline{S}M\overline{L}$, $\overline{S} \to \overline{S}M$; $\sigma_2 : L \to \overline{L}M$, $M \to L\overline{S}$, $S \to \overline{M}$, $\overline{L} \to \overline{M}L$, $\overline{M} \to S\overline{L}$, $\overline{S} \to M$. Thus, $\overline{XY} = \overline{Y}\,\overline{X}$ and $\overline{\overline{X}} = X$ $\forall X, Y$. The summary composition matrices read within basis $\{L, \overline{L}, M, \overline{M}, S, \overline{S}\}$:

$$M_1 = \begin{pmatrix} 1\,1 & 1\,0 & 0\,0 \\ 1\,1 & 0\,1 & 0\,0 \\ 1\,0 & 0\,1 & 1\,0 \\ 0\,1 & 1\,0 & 0\,1 \\ 0\,0 & 0\,1 & 1\,0 \\ 0\,0 & 1\,0 & 0\,1 \end{pmatrix}, \qquad M_2 = \begin{pmatrix} 0\,1 & 1\,0 & 0\,0 \\ 1\,0 & 0\,1 & 0\,0 \\ 1\,0 & 0\,0 & 0\,1 \\ 0\,1 & 0\,0 & 1\,0 \\ 0\,0 & 0\,1 & 0\,0 \\ 0\,0 & 1\,0 & 0\,0 \end{pmatrix}. \tag{1}$$

M_1 is primitiv, M_2 only irreducible. The characteristic polynomials $P_1 = m_\eta \cdot (x^3 - x + 1)$, $P_2 = m_\lambda \cdot (x^3 + x^2 - 2x - 1)$ yield PERRON-FROBENIUS (PF) eigenvalues η resp. λ; all PF eigenvectors are $(\vartheta, \vartheta, \lambda, \lambda, 1, 1)^{(t)}$. We use $|L| = \vartheta$, $|M| = \lambda$, $|S| = 1$, $|\overline{X}| = |X|$ and get $|\sigma_1(X)| = \eta\,|X|$, $|\sigma_2(X)| = \lambda\,|X|$ $\forall X$.

2.1. Non-empty Limit Translation Modules

Words occure with fixed frequence within each infinite sequence. So we conclude that the translation module $<\{\, t \in \mathbb{R} \mid \exists x : t + w_x = w_{t+x} \,\}>_{\mathbb{Z}}$ (w_x denoting arbitrary sub-words of length $\geq r$ at position x and $< \ldots >_{\mathbb{Z}}$ the $\mathbb{Z}$-span) contains at least the translations $|\sigma_i^n(w)|$, n large enough and $w = L\overline{M}SL, LM\overline{L}, \overline{L}L, L$ resp. $L\overline{S}\,\overline{M}\,\overline{M}L, L\overline{M}\,\overline{M}L, L\overline{S}\,\overline{M}L, L\overline{M}L, L\overline{S}L, LL, L$, for these words do occure within $\sigma_1^m(X)$ resp. $\sigma_2^m(X)$ $\forall X$, $m \geq 6$. Moreover, they cover the sequences. Next, we have $\vartheta = \eta(\lambda - 1) = \lambda(\vartheta - 1)$, $\lambda = \eta(\vartheta - 2\lambda + 2) = \lambda \cdot 1$, and $1 = \eta(-\vartheta + 2\lambda - 1) = \lambda(-\vartheta + \lambda + 1)$. Hence, as the lengths $|\sigma_i^n(w)|$ redecompose mutually to $|\sigma_i^n(X)|$, they do so downto $|L|$, $|M|$ and $|S|$; i.e., the translational modules of lengths $|L|$, more generally of $\eta^n|L|$ resp. $\lambda^n|L|$ are all $\mathbb{Z}[\vartheta]$ (they can't be larger). But all sub-words w are found within $|\sigma_i^n(w')|$, n appropriate and w' one of the mentioned words. This shows that the translational module is independent of r. Hence, the limit translation modules ($r \to \infty$) are $\mathbb{Z}[\vartheta]$ as well. Esp., they do clearly not vanish.

2.2. *The* BRAGG *Intensities*

The previous step tells something about correlations and hence on the FOURIER module (the support of possible BRAGG peaks), but it tells nothing about magnitudes of attached intensities. These are considered now. Using the notation $\rho_n^{(X)}$: DIRAC scatterer of n^{th} approximant, $z_j^{(n)}(X,Y)$: distance from first letter to j^{th} Y within $\sigma^n(X)$, θ : PF eigenvalue of the substitution matrix M, $M^n = (m_{X,Y}^{(n)})_{X,Y}$, we get as FOURIER transform

$$\begin{aligned}\left(\widehat{\rho}_{p(n+1)}^{(X_1)}(k)\right)_{X_1} &= \left(\delta_{X_1,X_2}\prod_{r=0}^{n}\exp(-ik\theta^{pr}z_1^{(p)}(X_1,X_2))\right)_{X_1,X_2} \cdot \\ &\cdot \left(\sum_{j=1}^{m_{X_2,X_3}^{(p)}}\exp(ik\theta^{pn}z_j^{(p)}(X_2,X_3))\right)_{X_2,X_3} \cdot \\ &\cdot \left(\delta_{X_3,X_4}\prod_{r=0}^{n-1}\exp(ik\theta^{pr}z_1^{(p)}(X_3,X_4))\right)_{X_3,X_4} \cdot \left(\widehat{\rho}_{pn}^{(X_4)}(k)\right)_{X_4},\end{aligned} \tag{2}$$

where p is to be choosen such that $m_{X,X}^{(p)} \geq 1 \ \forall X$ and $\delta_{i,j}$ is the KRONECKER symbol. The diagonal matrices reflect the demand that $\mathrm{supp}(\rho_{pn}^{(X)}) \subset \mathrm{supp}(\rho_{p(n+1)}^{(X)})$. The non-diagonal matrix of phase shifts we denote by $F_{pn}(k)$. Here we get with $x = \exp(ik\theta^n|X|)\ \forall X$ ($\theta = \eta^2$ resp. λ^2):

$$F_{2n,1}(k) = \begin{pmatrix} 1+\ell^2m+\ell^3m^2s & \ell m+\ell^4m^3s & \ell & \ell^3m+\ell^4m^2s & \ell^3m^2 & 0 \\ 1+\ell^3m^2s & \ell m+\ell^2m^2s+\ell^4m^3s & \ell+\ell^2ms & \ell^4m^2s & 0 & \ell^2m \\ 1 & \ell m+\ell^2m^2s & \ell+\ell^2ms & \ell^3m^2s & \ell^3m^3s & \ell^2m \\ ms+\ell m^2s^2 & \ell^2m^3s^2 & s & \ell ms+\ell^2m^2s^2 & \ell m^2s & 1 \\ 0 & ms & s & \ell ms & \ell m^2s & 1 \\ ms & 0 & s & \ell ms & \ell m^2s & 1 \end{pmatrix},$$

$$F_{2n,2}(k) = \begin{pmatrix} m+\ell m & 0 & 0 & 1 & 0 & \ell^2m \\ 0 & s+\ell s & \ell^2s & 0 & 1 & 0 \\ 0 & 1 & \ell+\ell m & 0 & 0 & 0 \\ m^2 & 0 & 0 & 1+m & 0 & 0 \\ 0 & s & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & \ell \end{pmatrix}. \tag{3}$$

From the normalisation $\widehat{\rho}_n^{\circ(X)}(k) = \widehat{\rho}_n^{(X)}(k)/\sum_Y m_{X,Y}^{(n)}$, i.e. $\widehat{\rho}_n^{\circ(X)}(0) = 1\ \forall n \in \mathbb{N}$, follows[2] that $\lim_n \widehat{\rho}_n^{\circ(X)}(k) = 0$ whenever $Abs(F_{p(n+m)}(k)\cdots F_{pn}(k))$ does not converge for every finite m (Abs applies to the entries). On the other hand, using results of number theory, we get: $\exp(ik\theta^n)$ converges iff θ is a PV number (and additional assumptions on k). Neither η^2 nor λ^2 is such. Thus, all intensities of the *infinite* sequences vanish relative to $k = 0$.

2.3. *High-dimensional Embedding*

To understand this fact better we use the high dimensional embedding $v_{\|} \leadsto v^{\Uparrow} = (v_{\|}, v_{\perp 1}, \ldots)$, where $(\ldots)$ are CARTESian co-ordinates: $L^{\Uparrow} = (\vartheta, -1, -\lambda)$, $M^{\Uparrow} = (\lambda, \vartheta, 1)$, $S^{\Uparrow} =$

$(1, -\lambda, \vartheta)$. Thus, we lift into $(\vartheta + \lambda - 1)\mathbb{Z}^3$. The interior space has twice the dimension of the physical space, it decomposes hence into a cross-product: The physical space scaling factors of $\eta = |\sigma_1(L)|/|L|$, $\lambda = |\sigma_2(L)|/|L|$ transform into the scaling factors $((2\vartheta + \lambda)/\vartheta, 2 - \vartheta, (2\lambda - 1)/\lambda)$, resp. $((\vartheta + \lambda)/\vartheta, 1 - \vartheta, (\lambda - 1)/\lambda)$; those are the roots of m_η, resp. m_λ. But, being no PV numbers, only one component of the interior space will be contracted, as in usual quasiperiodic cases, the other one gets larger! So, the «acceptance domain» (AD) must be unbounded. This does not contradict a finite density of sites in real space, if the contracting space component gets thin enough. This shall be considered now:

2.4. Dimensions

Lifting the inclusion relations of the substitution rules into the contracting space component we end with coupled iterated functions systems (CIFS) for that component of the «windows» or «AD's of the tiles». (Note, these terms are borrowed from the usual quasiperiodic setup, where thus constructed domains prove to be AD's for cut-and-project scenario, but keep in mind the problems at boundaries!) In here, the scaling factor θ_0 is the root smaller than one, therefore the (compact) attractors of the CIFS are unique. If we look for the dimensions of these attractors, the coupling of the CIFS (the matrices M_i are at least irreducible and X is the mirror image of $\overline{X}$) guarantees that the dimensions of the windows are the same. Thus, by applying the HAUSDORFF (H) measure on the equations for the attractors, we finally get

$$c \cdot (|\theta_0|^D M - \mathbf{1}) = 0, \tag{4}$$

$\mathbf{1}$ being the identity matrix, c the vector of H-volums and D the H-dimension. By re-reading this equation we observe that $1/|\theta_0|^D$ has to be the PF eigenvalue θ, i.e.,

$$D = -\log\theta / \log|\theta_0|. \tag{5}$$

This is true if no overlap exists. Else we use box counting dimension Δ and get $D \leq \Delta \leq -\lim_n \log \sum_X m_{X,Y}^{(n)} / \log|\theta_0|^n = -\log\theta/\log|\theta_0|$. This yields numerical values of $D_1 = 0.73675$, resp. $D_2 \leq 0.72736$. Thus, the attractors (themselves, not merely the boundaries!) are CANTOR sets; i.e., the «windows» have no LEBESGUE interior, they consist of boundaries only! Finally, by neglecting the term producing probable overlap, we get a lower bound too: $D_2 \geq -\log\tau/\log|\theta_0|$ (τ : golden ratio).

3. Extension to 2D Tilings

Now we show that these two sequences are not of selfinterest only. Both are 1D cross-sections of 2D tilings, built by triangles whose vertices belong to a regular heptagon. The extension of σ_1 was already preliminarily investigated by NISCHKE and DANZER[1] (ND) and is hence out of the scope here. For the 2^{nd} we label the edges of triangle A with (M, S, S), B with $(L, S, \overline{M})$, C with $(L, L, \overline{S})$, D with (L, M, M), and P with $(L, \overline{M}, S)$, each in both enantiomers. Edge-tupels are to be read as circuits. The length of an edge is thus obvious, the tiles well-defined. As σ_2 applies to the edges, it defines an unique 2D decomposition. For convenience we give the summary rule too: $\Sigma : A \rightarrow \{\overline{A}, B\}$, $B \rightarrow \{\overline{C}, P, \overline{P}\}$, $C \rightarrow \{\overline{C}, D, \overline{P}\}$, $D \rightarrow \{\overline{B}, C, D, \overline{D}\}$, $P \rightarrow \{A, \overline{B}, \overline{D}\}$. The matrix (e.g. in the basis $\{A, \overline{A}, B, \overline{B}, C, \overline{C}, D, \overline{D}, P, \overline{P}\}$) is primitiv and its characteristic polynomial reads $P = (x^2 + x + 1)^2(x^3 + x^2 - 2x - 1)(x^3 - 5x^2 + 6x - 1)$; the PF eigenvalue is λ^2, the correspoding eigenvectors are $(1, 1, \vartheta, \vartheta, \eta, \eta, \lambda\vartheta, \lambda\vartheta, \vartheta, \vartheta)^{(t)}$, describing the relative frequencies as well as the relative volumes of the tiles. But the directions X and $\overline{X}$ are antiparallel $\forall X$, hence we get only 7-fold quasisymmetry. I.e., its «AD» is the direct sum of a whole plane with a

7-fold fractal (empty interior!). The ND tiling yields an «AD» which is the sum of a plane with a 14-fold fractal (without interior). The fractal components are depicted below, using about 3000 lifted points. We embed into the lattice A_6.

In addition to the decompositions, composition rules do exist for both tilings. Moreover, all of these procedures can be locally defined on bare tilings. The ND tiling posseses perfect matching rules[1]. The other one definitively not.

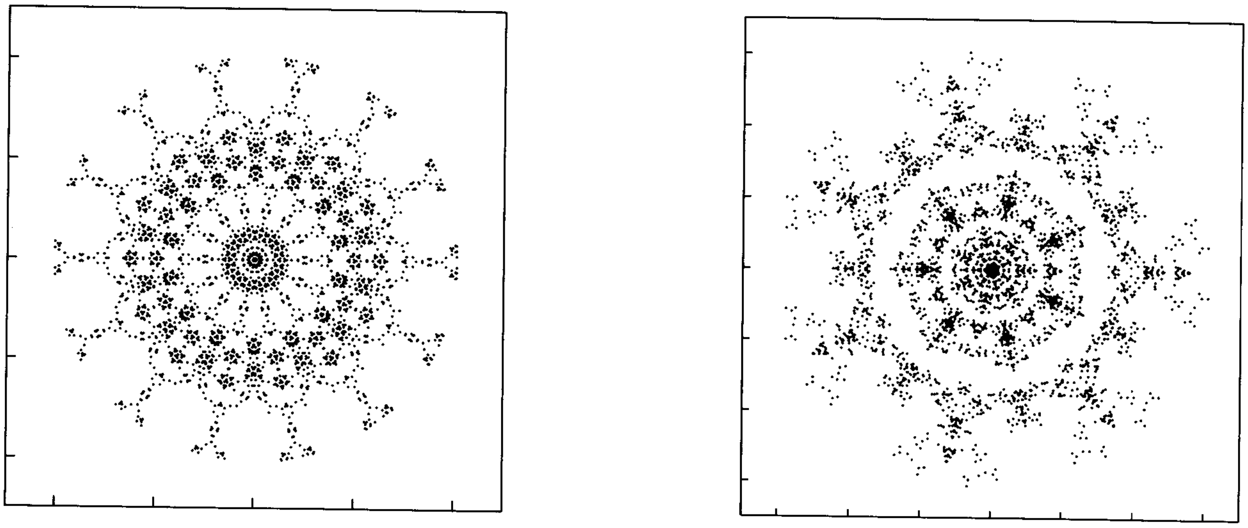

Figure 1: To the left a lift of the ND tiling into the 1^{st} internal component, to the right that of Σ into the 2^{nd} is shown. Lifts of σ_1 resp. σ_2 correspond to horizontal diametral cross-sections.

4. Conclusions

We have given two examples of non-PV substitutional sequences (resp. tilings). The non-PV property implies the disappearance of the (non-trivial) BRAGG components of the FOURIER spectrum. On the other hand the support of such peaks normally is given (via PATTERSON) by the reciprocal of the limit translational module. Thus, our examples are somehow counter-intuitiv, and show how few is known for general substitutional sequences.

Here, we have investigated the BRAGG part only. But it should be added that other spectral researches conjecture the intensity measure of non-PV substitutional structures to be purely singular continuous[3].

Another remark to eq. (4): It provides an extension of the connection between volumes in internal space and frequencies of occurence to non-integral dimensions.

References

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