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PERFECT MATCHING RULES FOR UNDECORATED TRIANGULAR TILINGS WITH 10-, 12-, AND 8-FOLD SYMMETRY

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Perfect matching rules are derived for quasiperiodic triangular tilings with 10-, 12-, and 8-fold symmetry. We use the composition/decomposition approach via local inflation/deflation symmetry and emphasize the locality of our procedure: The matching rules given here are formulated using certain decorations of the tilings. These decorations turn out to be redundant, i.e., they are locally derivable from the undecorated tilings. Hence, the latter are determined by perfect matching rules as well.

1. Introduction

The experimental discovery of quasicrystals with icosahedral symmetry[1] and, simultaneously or only a little later, with twelvefold,[2] tenfold,[3] and eightfold symmetry[4] has not only established a new branch of solid state physics but also given an enormous impact on the theory of tilings. The latter proves useful for the description of quasicrystals[5,6] and many examples have been worked out in detail.

Perhaps amongst the most fascinating properties of these ideal tilings are the existence of *inflation/deflation symmetries* — which are useful for finding a consistent indexing scheme of the Bragg reflection peaks — and of *matching rules* that enforce aperiodicity (in which case they are called *strong*) or specify uniquely a single local isomorphism class (in which case they are called *perfect*), compare Ref. 7. Already the well-known Penrose tiling (e.g., in its version with rhombi) exhibits both phenomena: a deflation and an inflation is known that can be formulated locally, and also perfect matching rules that make the two Penrose rhombi to an aperiodic set, cf. p. 542f of Ref. 8. In the sequel, other tilings with perfect matching rules were found, e.g., Refs. 9–13, and a hierarchy of matching rules has been formulated, compare Refs. 14, 7, 15.

Obviously, tilings cannot be taken literally for pictures of the microstructure of quasicrystals. Like unit cells in the crystalline case, they may provide a convenient representation of the global order of a given structure which in turn may arise as a decoration of a certain tiling. For such a quasicrystalline structure, several different tilings may be equally well suited. This observation leads to the concept of mutual local derivability[16] which gives an equivalence condition for tilings and patterns in a general context. In particular, quasiperiodic structures are covered therein.

Roughly speaking, two given tilings are locally equivalent (*s*ymmetry preserving *m*utually *l*ocally *d*erivable, SMLD), if each of them can be reconstructed from the other one just by inspection of patches up to a certain, *a priori* determined, finite size. This notion immediately extends to equivalence of whole classes of *l*ocal *i*somorphism (LI classes). A tiling (or, more generally, a pattern) describes the global order of a physical structure (e.g., a set of positions of atoms), if it is locally equivalent to the latter. In view of this equivalence concept, one has to distinguish sharply between tilings and certain decorations of them. In particular, the formulation of the matching rules of the Penrose tiling mentioned above is usually presented with the help of certain arrows placed at the boundaries of the rhombi. It turns out that this decoration is locally equivalent to the undecorated pattern (one has to inspect only the nearest and next to nearest neighbours of vertex points). Therefore, also the LI class of *undecorated* Penrose tilings possesses perfect matching rules. As a consequence it is possible to construct local interactions which stabilize physical structures described — in the above sense — by Penrose tilings.

The situation is different in the case of the eightfold symmetric Ammann-Beenker tiling[17,11,8,9,10] and the twelvefold symmetric Socolar tiling[9] or the locally equivalent Niizeki-Mitani-Gähler tiling[10]: perfect matching rules have been given in the articles mentioned. They are formulated by means of certain decorations, but the latter *cannot* be derived locally from the undecorated patterns in either case; moreover, it has been shown that matching rules for the *undecorated* patterns do not exist. In a recent publication,[18] it is very confusing that there is no distinction at all between decorated and undecorated patterns. This distinction is necessary from the crystallographic point of view because the patterns belong to different SMLD classes. So it remained an open problem to establish perfect matching rules for natural undecorated tilings with eight- or twelvefold symmetry. Possible candidates were known for some time[19] but the connection to the matching rule problem was realized only recently by the relation of the various tilings to those with known matching rules in terms of local derivability.[20,21] It is the main goal of this article to establish perfect matching rules for certain undecorated patterns with twelvefold and eightfold symmetry.

Before we continue, let us give a brief survey on how this article is organized. The following section explains the techniques used to obtain such rules for the triangular tilings $\mathcal{T}^{\star}_{A_4}$, $\mathcal{T}^{(12)}_{D_4}$, and $\mathcal{T}^{(8)}_{D_4}$. Thereafter, Secs. 3–5 deal explicitly with these patterns. In each case, *local* inflation/deflation rules are presented using locally derivable decorations. (Note that the existence of such *local* inflation/deflation rules is a non-trivial property.) Either of our tilings can be derived by the projection method (cf. Ref. 23 and Appendices C and B of Ref. 19, respectively). We use, as natural ansatz for matching rules, vertex configurations (with an additional decoration which is shown to be locally derivable). Furthermore, these rules are contracted to interactions of pairs of tiles by means of decorations of vertices and edges.

In particular, Sec. 3 deals with the tenfold symmetric triangle pattern $\mathcal{T}^{\star}_{A_4}$. (The basic definitions and properties of this tiling can be found in Refs. 22, 23. Its relation to binary tilings with perfect matching rules will be discussed in a forthcoming publication.[24]) In Sec. 4, the analogous program is carried out for the twelvefold symmetric triangular tiling $\mathcal{T}^{(12)}_{D_4}$ that has briefly been described in Appendix C of Ref. 19. It will be shown additionally that this tiling contains the local information of the undecorated Niizeki-Mitani-Gähler tiling[25,26] *and* its matching rule decoration,[10] which comes as a little surprise. In the Appendix, we describe a tiling with twelvefold symmetry which allows the local derivation of matching rules also from the set of vertex sites alone. Section 5 is the obvious application of the same technique to the eightfold symmetric triangular tiling $\mathcal{T}^{(8)}_{D_4}$, compare Appendix B of Ref. 19. It turns out that this tiling contains the complete local information of the Ammann-Beenker pattern[11] *including* its matching rule decorations, cf. p. 556f of Ref. 8. It thus resembles the situation of Sec. 4. This discussion is followed by some concluding remarks in Sec. 6.

2. The Techniques

Before we present the explicit examples we will briefly describe the methods used. We follow those outlined on pages 558ff of Ref. 8. They are used in Refs. 10, 18 too.

Let $\mathcal{T}$ be a tiling in n-dimensional space. An R-patch ($R \geq 0$) of $\mathcal{T}$ surrounding a point q is the set of all elements of $\mathcal{T}$ which intersect the R-ball around q. (For $R = 0$, this set consists of those tiles which cover q in an inner part or within any lower-dimensional boundary.) Two tilings $\mathcal{T}_1$ and $\mathcal{T}_2$ are said to be *locally isomorphic* (with respect to Euclidean motions) if, for arbitrary $R \geq 0$, every R-patch of $\mathcal{T}_1$ can be transformed into an R-patch of $\mathcal{T}_2$ by a Euclidean motion, and vice versa. (As is well-known, this does not imply the existence of a motion which brings the two tilings to global coincidence.) The set of all tilings that are locally isomorphic to a given tiling $\mathcal{T}$ is called the local isomorphism (LI) class of $\mathcal{T}$. For most physical applications, it is reasonable to consider the members of an LI class as equivalent.

The definition of LI classes involves the inspection of patches of finite, but arbitrary size. Of special interest are those LI classes which can be determined using only patches of a fixed finite size, i.e., which possess perfect matching rules. Formally, a *matching rule* with radius R_{mr} is a list of R_{mr}-patches; a tiling $\mathcal{T}$ fulfills this rule if every R_{mr}-patch of $\mathcal{T}$ can be found in the given list. Such a matching rule is said to be *perfect* if there exists a tiling fulfilling it and if, on the other hand, all tilings which fulfill it belong to the same LI class of tilings.

To decide whether some given matching rule is perfect is a fairly complicated problem, compare Ref. 15. The situation is much simpler in the case where the LI class of tilings under consideration possesses an inflation/deflation symmetry. This means that there is a similarity transformation Θ of dilation factor $\vartheta > 1$ such that for some (and therefore every) element $\mathcal{T}$ of the LI class there exists an element

$\mathcal{IT}$ of the LI class of $\Theta\mathcal{T}$ which is locally equivalent to $\mathcal{T}$ (i.e., belongs to the same SMLD class as $\mathcal{T}$, cf. Ref. 16). The local equivalence of $\mathcal{T}$ and $\mathcal{IT}$ means that $\mathcal{IT}$ is locally derivable from $\mathcal{T}$ and vice versa. In practice, this inflation and the converse deflation are defined locally by means of composition and decomposition, respectively. This locality demands the existence of a radius R_I such that for every Euclidean motion S and every point q the identity of the R_I-patches (around q) of $\mathcal{T}$ and $S\mathcal{T}$ implies identity of the 0-patches (around q) of $\mathcal{IT}$ and $S\mathcal{IT}$, and the existence of a radius R_D which provides the analogous relation interchanging $\mathcal{T}$ and $\mathcal{IT}$. A consequence of this condition is that every $(R + R_I)$-patch in $\mathcal{T}$ uniquely determines the R-patch around the same point in $\mathcal{IT}$, as well as every $(R + R_D)$-patch in $\mathcal{IT}$ uniquely determines the corresponding R-patch in $\mathcal{T}$.

Now, let us assume that we are given a certain LI class represented by a member $\mathcal{T}_0$, e.g., obtained by the well-known projection method from a higher-dimensional periodic structure. In particular, we assume that $\mathcal{T}_0$ is repetitive (as is always guaranteed for tilings generated by projection[16,27]). Every matching rule satisfied by $\mathcal{T}_0$ is characterized by its radius R_{mr} and consists of the set of all R_{mr}-patches of $\mathcal{T}_0$. Let us denote, by the symbol $\{\mathcal{T}_0\}_R$, the class of all tilings with the property that for a fixed R every R-patch of them can be found in $\mathcal{T}_0$ (i.e., can be transformed into an R-patch of $\mathcal{T}_0$ by Euclidean motions). Because of the repetitivity of $\mathcal{T}_0$, its LI class is precisely the intersection of all $\{\mathcal{T}_0\}_R$, $R \geq 0$. (Obviously, $R' \geq R$ implies $\{\mathcal{T}_0\}_{R'} \subseteq \{\mathcal{T}_0\}_R$.) In order to show that the LI class of $\mathcal{T}_0$ is determined by a matching rule with radius R_{mr}, it suffices to show that for every element $\mathcal{T}$ of the class $\{\mathcal{T}_0\}_{R_{mr}}$ it is true that every R-patch of $\mathcal{T}$ can be found in $\mathcal{T}_0$, i.e., $\{\mathcal{T}_0\}_{R_{mr}} \subseteq \{\mathcal{T}_0\}_R$ $(R \geq 0)$. This can be achieved if $\mathcal{T}_0$ has an inflation/deflation symmetry as explained above which fulfills some additional properties. First, the mapping $\mathcal{I}$ (the inflation transformation) must be extendable from the LI class to the whole class $\{\mathcal{T}_0\}_{R_{mr}}$ such that for every two elements $\mathcal{T}_1$ and $\mathcal{T}_2$ of this latter class, every point q, and every Euclidean motion S, an identity of the R_I-patches around q of $\mathcal{T}_1$ and $S\mathcal{T}_2$ implies an identity of the corresponding 0-patches in $\mathcal{IT}_1$ and $S\mathcal{IT}_2$. Second, the inverse $\mathcal{D}$ of $\mathcal{I}$ (the deflation transformation) fulfills the analogous condition with replacing R_I by R_D and $\mathcal{T}_0$ by $\Theta\mathcal{T}_0$; additionally, the relation $R_{mr} > (\vartheta/(\vartheta-1))R_D$ must hold. And third, the inflation transformation $\mathcal{I}$ must map $\{\mathcal{T}_0\}_{R_{mr}}$ into $\{\Theta\mathcal{T}_0\}_{\vartheta R_{mr}}$. The last condition is the nontrivial one whereas the other ones can be achieved for every inflation/deflation symmetry (having local transformations) for suitable radii R_D, R_I, and R_{mr}.

If these three conditions are fulfilled, the fact that the LI class of $\mathcal{T}_0$ is determined by a matching rule with radius R_{mr} is seen as follows. Let $\mathcal{T}$ be an element of $\{\mathcal{T}_0\}_{R_{mr}}$. Given the radius R and a point q arbitrarily, we have to find the R-patch (around q) of $\mathcal{T}$ in $\mathcal{T}_0$. We may choose n such that $R_{mr} \geq R/\vartheta^n + (\vartheta/(\vartheta-1))R_D$, then, $\vartheta^n R_{mr} \geq R + \sum_{i=0}^{n-1} \vartheta^i R_D$. The third condition implies that the $\vartheta^n R_{mr}$-patch around q of $\mathcal{I}^n\mathcal{T}$ can be found in $\mathcal{I}^n\mathcal{T}_0$, with motion S, say. Successive application of the second condition shows that the R-patch around q of $\mathcal{T}$ can be found in $\mathcal{T}_0$ using the same motion S.

In summary, one has proved a certain matching rule to be perfect for a given repetitive LI class of tilings if one has shown the following conditions to hold:

- There is a *local* inflation $\mathcal{I}$ for the given LI class which can be extended to the class of all tilings fulfilling the matching rule.
- There is a *local* deflation $\mathcal{D}$ which is the inverse of $\mathcal{I}$ and is also extendable to the class of all tilings fulfilling the matching rule; the matching rule radius must obey the inequality mentioned above.
- The properly rescaled matching rule is fulfilled by the inflated tilings.

In order to present explicit inflation/deflation transformations, things become simpler if the deflation radius R_D vanishes. Therefore, it is convenient to decorate these tiles in a suitable way, e.g., by arrows (as in the following sections), in order to remove ambiguities in the deflation of tiles; the matching rules we will formulate will apply only to the set of decorated tiles. In all three examples to follow the decorated tilings do belong to the same SMLD class, wherefore the existence of matching rules will be proved automatically also for the *undecorated* tilings.

3. The Decagonal Case

Let us start now with the decagonal tiling of the plane by the two golden triangles which can be derived from the four-dimensional root lattice A_4 by dualization and projection[23]; the LI class of these tilings will be denoted by $\mathcal{T}^{\star}_{A_4}$. The structure of the four-dimensional lattice provides a transformation which manifests itself as a rescaling by $1/\tau = \frac{1}{2}(\sqrt{5}-1)$ in the two-dimensional tiling space and results in a proper decomposition of the tiles into rescaled copies. One finds that for both the acute and the obtuse triangle two different decompositions occur. This ambiguity can be removed by the introduction of arrows which indicate the decomposition of a given tile according to Fig. 3.1.

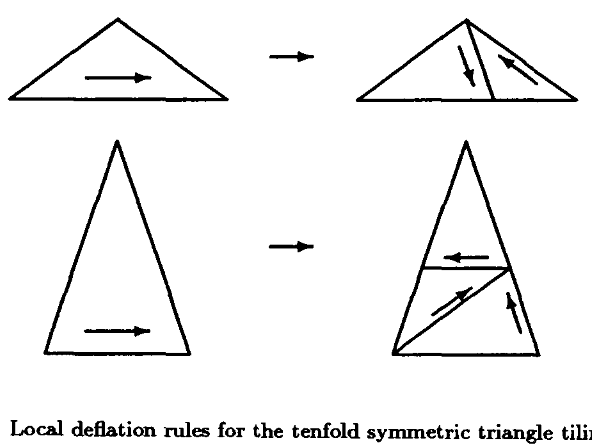

Fig. 3.1. Local deflation rules for the tenfold symmetric triangle tiling $\mathcal{T}^{\star}_{A_4}$.

By a closer inspection of the acceptance domains,[23] it is easily checked that the decoration of the rescaled triangles is forced by the global transformation mentioned above. Furthermore, the acceptance domains of the *decorated* triangles can be calculated. By the usual methods, one obtains the list of decorated vertex configurations that will occur in the decorated tiling as generated with the new acceptance domains; these configurations are found in Fig. 3.2. That the relation between decorated and undecorated tilings is a local equivalence, can be established using the rule for the derivation of the decorations depicted in Fig. 3.3. It can be shown that for every tiling of the LI class of $\mathcal{T}^{\star}_{A_4}$ this rule completely determines the decoration of all tiles. (The other direction is trivial, just erase all arrows.)

Now, having shown the local equivalence between decorated and undecorated tilings, we can restrict ourselves to the consideration of the *decorated* ones in order to establish perfect matching rules there. More specifically, we will show that every tiling of the plane consisting of decorated golden triangles such that the vertex configurations which occur can be found in Fig. 3.2 will belong to the LI class of tilings obtained by decoration of tilings of $\mathcal{T}^{\star}_{A_4}$ according to Fig. 3.3.

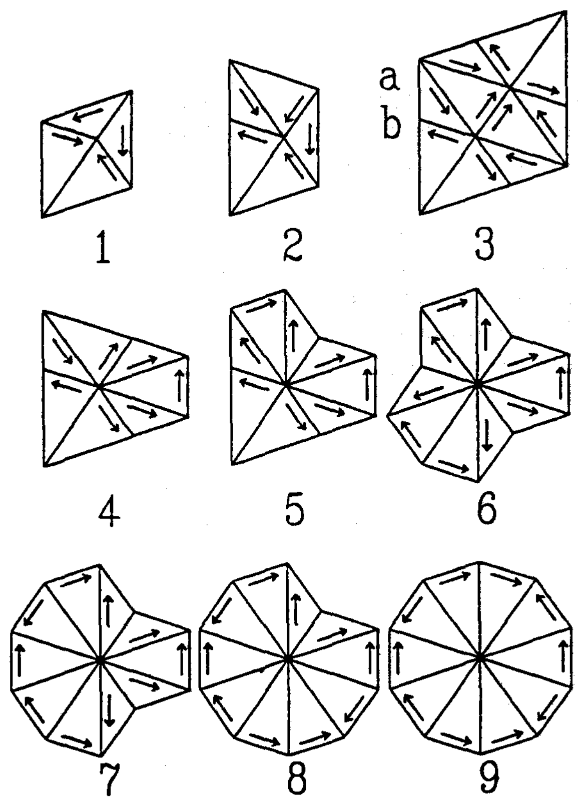

Fig. 3.2. The decorated vertex configurations of $\mathcal{T}^{\star}_{A_4}$.

For short, let us call regular the vertex configurations of Fig. 3.2, as well as decorated tilings containing only such configurations.

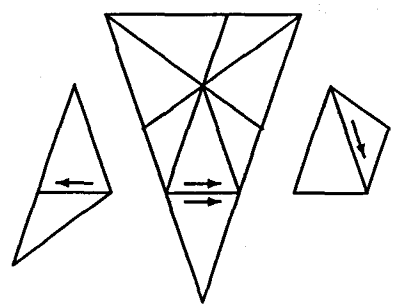

Fig. 3.3. Local decoration rules for $\mathcal{T}^{\star}_{A_4}$.

We have already established the existence of a local deflation transformation with $R_D = 0$. The inverse inflation transformation can be formulated as follows.

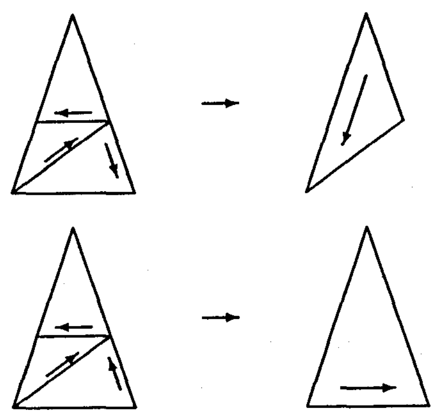

Fig. 3.4. Local inflation rules for $\mathcal{T}^{\star}_{A_4}$.

A short glance at the vertex configurations in Fig. 3.2 shows that in a regular tiling every obtuse triangle is surrounded in one of the two manners depicted on the left side of Fig. 3.4. It is easy to see that the composition rule indicated in Fig. 3.4 can be applied everywhere in any regular tiling without contradiction and indeed provides the inverse transformation of the above deflation rule. Furthermore, the result of such an inflation of a regular tiling will be a face-to-face tiling with vertex points only in positions of vertex points of the original tiling. According to the general remarks in Sec. 2, it remains to be shown that the inflation of a regular tiling leads to a regular tiling. The vertex configurations 1–7 are readily seen to inflate only into regular vertex configurations. However, for configurations 8 and 9, there is, in each case, one possibility of inflation which yields an irregular configuration (if there is no additional information from the surrounding). Nevertheless, at least all configurations in the inflated tiling where an arrow points towards the central vertex must be from the list of regular configurations. A bit of combinatorics shows that a patch as depicted in Fig. 3.5 will not arise by inflation.

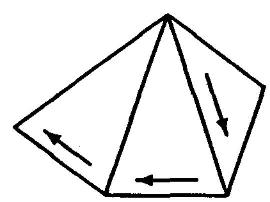

Fig. 3.5. An example for an irregular patch.

Only the irregular possibilities for 8 and 9 contain such patches. We see that the inflation of an infinite regular tiling cannot produce irregular vertex configurations and thus delivers a regular tiling. This proves the existence of a perfect matching rule with radius $R_{mr} = 0$. (Strictly speaking, for the argument explained in Sec. 2, we have to choose the radius slightly larger than 0, but, afterwards, as only vertex configurations matter, we may reduce the radius to 0.)

As a bonus of the projection method, the list of occurring decorated vertex configurations has provided, in quite a natural way, candidates for a matching rule. The formulation of deflation and inflation as rules that relate to single (decorated) tiles, makes it advantageous to have the matching rules in an analogous form. Honestly, we have to say that we do not have a *general* method of how to condense these rules into the form of simple decorations of the edges and vertices of tiles. Here we start with the observation that every vertex configuration will end after a finite number of deflations in the configuration with the number 9 in Fig. 3.2. If one allows only even-numbered steps of deflation, the orientation of this configuration will never change afterwards. The decoration of this limit configuration is now translated into a vertex decoration. One can therefore introduce an additional small tile as depicted in Fig. 3.6 (and corresponding gaps in the original tiles).

In general, such vertex decorations alone are not sufficient to reconstruct exactly the set of regular vertex configurations by the simple ansatz: *Tiles may join if and only if the additional vertex tile matches the corresponding gaps.* But introducing edge decorations, it becomes possible. Irregular configurations as those mentioned above seem to be allowed again, but the rules (Fig. 3.6) enforce, in such a case, a wrong vertex tile somewhere else, and thereby a contradiction. As an additional remark, let us mention that for the tiling $T^{\star}_{A_4}$ it can be shown that the edge deco-

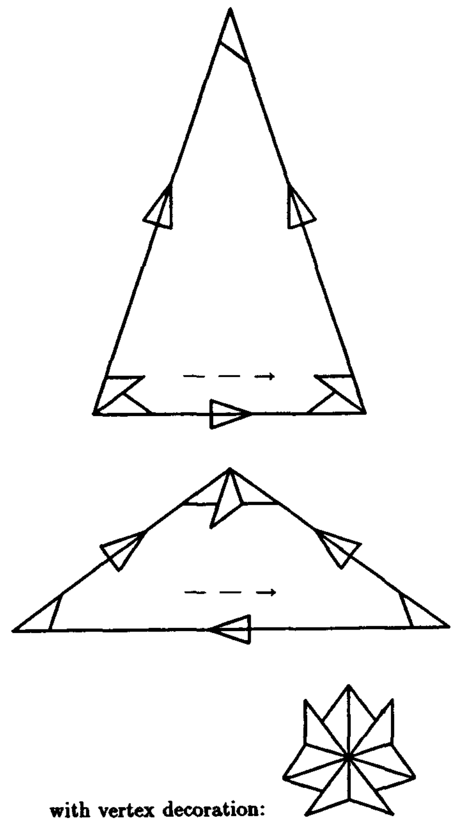

Fig. 3.6. Local matching rules for $\mathcal{T}^{\star}_{A_4}$. (The dashed arrows indicate the correspondence to the decoration rules of Fig. 3.3.)

rations might be chosen in the same shape and size as the additional vertex tiles, e.g., situated on those points where vertices will occur in case of deflation(s), and orientated in the same direction as the former edge decoration. Then, the additional edge tiles become identical to the vertex tiles. (In such a case, one has to show that such tiles cannot occur partly as edge tiles and partly as vertex tiles.) If one regards the decorations of vertices and edges as introduction of additional tiles, this reduces the number of necessary tiles to three (including the modified triangles). Let us emphasize again that the structure obtained this way belongs to the same SMLD class as the original tiling $\mathcal{T}^{\star}_{A_4}$.

4. The Dodecagonal Case

Quite recently, two dodecagonal triangular tilings (as well as octagonal ones) have been derived during the investigation of the root lattice D_4, see Ref. 19. Unfortunately, the tilings needed here (taking the projection of the Delaunay cells as acceptance domains) have been explained in much less detail than the dual cases with the projection of the Voronoi cell in perpendicular space. Nevertheless, we have no doubts that it will be quite easy for the reader to reconstruct missing information.

Let us start with the dodecagonal case. The acceptance domains for vertices of $\mathcal{T}_{D_4}^{(12)}$ are the projections of the three Delaunay cells. All of them have the outer shape of a square and they are rotated copies of one another. The three resulting classes of vertex configurations, whose elements are rotated copies as well, need therefore not be distiguished. The tiling consists of four triangles, see Fig. 4.1, and has the special property that it is separated by the acute ones. (By separation we mean that the set of edges of all those acute triangles is equal to the set of edges of all the other triangles and thereby already the complete set of edges of the entire tiling.)

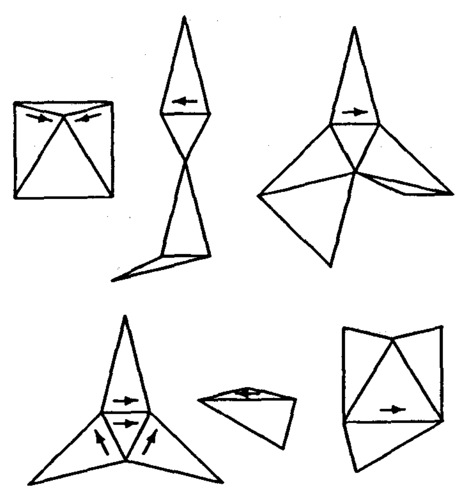

Fig. 4.1. Local decoration rules for the twelvefold symmetric triangular tiling $\mathcal{T}_{D_4}^{(12)}$. (One has to decorate the small triangles at last.)

As in the case of $\mathcal{T}_{A_4}^{\star}$, one has to decorate $\mathcal{T}_{D_4}^{(12)}$ in a *local* manner first (Fig. 4.1) in order to get local deflation (Fig. 4.2) and inflation rules (Fig. 4.3). It is due to the separation property that the somewhat uncommon deflation (decomposition) into patches of slightly deformed outer shapes produces neither gaps nor overlaps. The inflation rule becomes very simple as a result of this separation property: one has to invert only the deflation rule of the acute triangle. In quite an analogous way one checks that this set of local rules respects the LI class. Just as before, decorated vertex configurations are used for matching rules, and one proves that they fulfill the conditions mentioned above. The given deflation and inflation correspond to a dilation by $\vartheta = \sqrt{\varrho}$, $\varrho = 2 + \sqrt{3}$, followed by a rotation of 15^o.

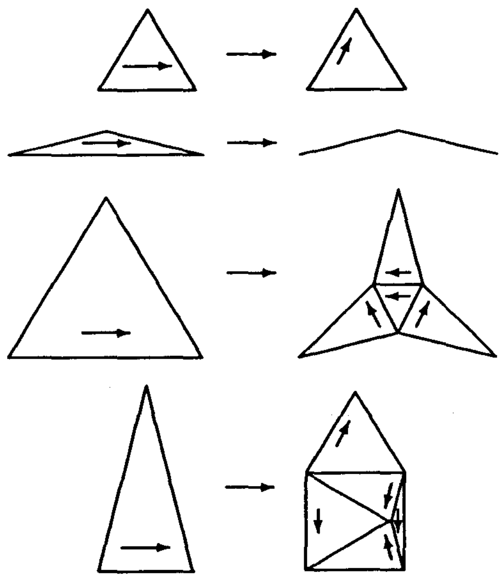

Fig. 4.2. Local deflation rules for $\mathcal{T}_{D_4}^{(12)}$.

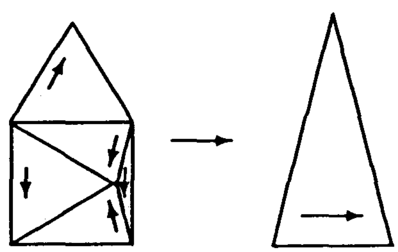

Fig. 4.3. Local inflation rules for $\mathcal{T}_{D_4}^{(12)}$. (This is already sufficient due to the separation of this tiling. The decoration of the other inflated triangles is to be read from Fig. 4.1.)

Finally, one puts the matching rules in a similar form as in the previous section by deriving a decoration of edges and vertices (Fig. 4.4). Decorations for edges are even simpler in this case, due to the separation. But the way we found vertex decorations in the previous section does not apply to this case. This is due to the fact that $\mathcal{T}_{D_4}^{(12)}$ has no symmetry preserving vertex configuration. The one in which the even deflations end consists of four acute triangles with big ones in between. (Only for those acute ones that derivation would apply.) A vertex decoration, which wants to become a "matching rule", has to be compatible with the set of regular vertex configurations. In order to make the vertex decoration compatible with the regular configurations one has to divide each such configuration in a symmetry-preserving manner, i.e., in this case, in twelve equal sectors. Next, we choose one

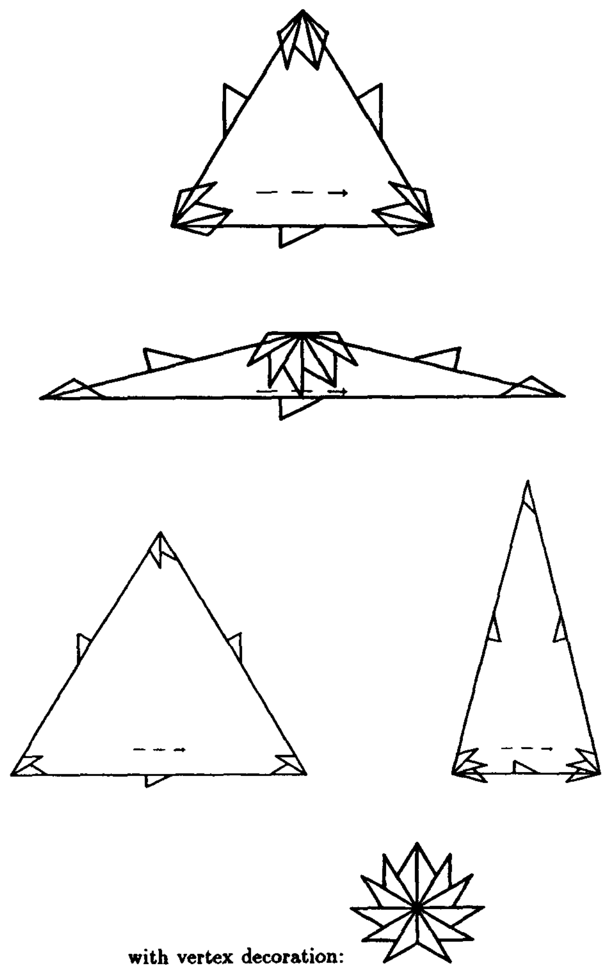

Fig. 4.4. Local matching rules for $\mathcal{T}_{D_4}^{(12)}$. (The dashed arrows indicate the correspondence to the decoration rules of Fig. 4.1.)

orientation of one sector in one of the vertex configurations, for instance the one given by the previous method for the acute triangles. Then, we orientate all the sectors of all configurations which are in the same position to that (decorated!) tile.

Moreover, we orientate the opposite sector in the opposite sense (with respect to rotation). This is not only a mere analogy of what has been produced in the former section, since all decorated vertex configurations show such a behaviour (i.e., if for any tile in a configuration a reflected tile appears as well, then the decoration is the reflected one, too). Applying these rules until all sectors of all regular configurations are decorated by orientation, one can now introduce, just as before, an additional vertex tile which reflects exactly that orientation of sectors.

This ansatz is a generalization of the one of Sec. 3, for it produces, applied to $\mathcal{T}^{\star}_{A_4}$ (or other applicable tilings), exactly the same vertex decoration. The above derivation has been applied to the tiling $\mathcal{T}^{(12)}_{D_4}$ as well as to $\mathcal{T}^{(8)}_{D_4}$ (in the next section).

In the twelvefold case, there appears the additional fact that the orientations are not specified uniquely. Only eight of the twelve sectors are dependent of each other, as well as the other four (this resembles the configuration of the vertex mentioned in which four acute triangles are separated by big ones). Thus it is possible to define different vertex decorations, applying to all regular vertex configurations simultaneously. Here, we have chosen the symmetric one. Again, one proves that these decorations of edges and vertices respect the whole LI class (by reproducing exactly the set of regular configurations). Whether the second choice results in equivalent findings has not been analyzed.

We end this section with a remark on the ***N***iizeki-***M***itani-***G***ähler tiling, $\mathcal{T}_{NMG}$, independently found in Refs. 25 and 26, and its matching rules.[10] This tiling is locally derivable from $\mathcal{T}^{(12)}_{D_4}$ by taking the circumcenter of the acute triangles as vertices. The edges will be derived from them by drawing all shortest distances between points. Looking at the acceptance domains of these two tilings, one can show that the LI class of the Niizeki-Mitani-Gähler tiling includes a globally threefold symmetric version, but the LI class of $\mathcal{T}^{(12)}_{D_4}$ does not; therefore one cannot find a symmetry preserving, local derivation rule from $\mathcal{T}_{NMG}$ to $\mathcal{T}^{(12)}_{D_4}$. (Reference 16 shows, moreover, that one even cannot find a symmetry breaking one. This is due to the fact that the boundaries of the acceptance domains of edges or tiles from $\mathcal{T}^{(12)}_{D_4}$ include angles of $2\pi n/24$ (n odd), whereas those of $\mathcal{T}_{NMG}$ require only even n's, i.e., the latter domains cannot reconstruct — symmetry breaking or not — the former.)

As already mentioned in the Introduction, the decoration of $\mathcal{T}_{NMG}$ needed for matching rules break the global symmetry of the pattern and so cannot be local. After having broken the symmetries, the question of local equivalence of the *decorated* $\mathcal{T}_{NMG}$ with $\mathcal{T}^{(12)}_{D_4}$ re-arises. We want to show now that the answer is positive. The first direction is quite easy, as seen in Fig. 4.5. The cross-shaped tile indicates the vertex decoration of $\mathcal{T}_{NMG}$. In order to get the edge decorations, derivations from the orientation of the decorated great and obtuse triangles of $\mathcal{T}^{(12)}_{D_4}$ can be given just as easily: these decorations depend on the acute triangles, as shown in Fig. 4.1, and the edge decorations are locally dependent on the vertex decorations. The inverse case is seen in an even simpler way due to the fact that $\mathcal{T}^{(12)}_{D_4}$ is separated by the acute triangles and therefore only those need to be reconstructed. This is done by inversion of Fig. 4.5. (The relative size of the triangle is given by the ratio between the longer edges of $\mathcal{T}^{(12)}_{D_4}$ and those of $\mathcal{T}_{NMG}$ which is $\sqrt{\varrho/2}$.)

It has been shown in this section that the (undecorated) tiling $\mathcal{T}^{(12)}_{D_4}$ is locally equivalent to the decorated one, and that the *decorated* tiling $\mathcal{T}_{NMG}$ (which is inequivalent to the undecorated $\mathcal{T}_{NMG}$) is equivalent to them as well. Thus, one has two independent proofs for perfect matching rules of this SMLD class.

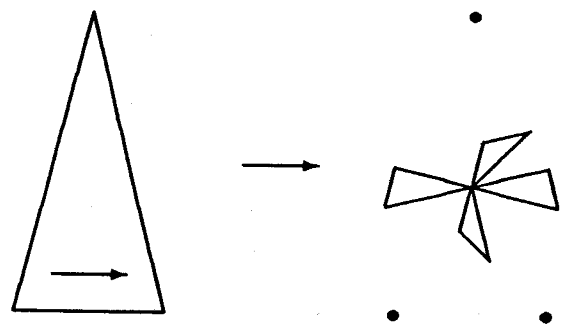

Fig. 4.5. The derivation of the vertex decoration of $\mathcal{T}_{NMG}$ from $\mathcal{T}_{D_4}^{(12)}$.

At this point, we would like to mention that the tiling $\mathcal{T}_{D_4}^{(12)}$ cannot be reconstructed from the vertex sites alone: one also needs the positions of the acute triangles. However, also a tiling does exist in the SMLD class of $\mathcal{T}_{D_4}^{(12)}$ which *can* be reconstructed from the vertex sites alone. It also has perfect matching rules. Such a tiling is described in the Appendix.

5. The Octagonal Case

Like $\mathcal{T}_{D_4}^{(12)}$, the tiling $\mathcal{T}_{D_4}^{(8)}$ is derived from D_4. In this case, one uses just another projection direction. Due to this fact, the shapes of the three acceptance domains are different. Two of them still remain equal-sized squares, but the third one becomes an octagon (with the same incircle radius). Therefore, one has to distinguish two of the three vertex classes from the third one.

It is by chance that the acceptance domain of the well-known ***A***mmann-***B***eenker tiling $\mathcal{T}_{AB}$ (see Ref. 11) has exactly the same (furthermore likewise orientated) outer octagonal shape as one of the vertex classes of $\mathcal{T}_{D_4}^{(8)}$. That is, this vertex class, taken for itself, forms $\mathcal{T}_{AB}$. It might be worth noticing that this vertex class of $\mathcal{T}_{D_4}^{(8)}$ is locally distinguishable from the others (it includes exactly those vertices which contain the tops of the acute or of the obtuse triangles). That means that the tiling $\mathcal{T}_{AB}$ is locally derivable from $\mathcal{T}_{D_4}^{(8)}$, but is not equivalent to it (the proof runs in the very same way as in the dodecagonal case).

As in the decagonal and the dodecagonal cases, one has a locally derivable decoration (Fig. 5.1) which enables one to define a local deflation (Fig. 5.2) and inflation (Fig. 5.3) in a unique manner.

It follows directly from the vertex configurations corresponding to the square domains that the middle-sized edges are situated either between an acute and an obtuse triangle or between an acute one and an oblique one. It is therefore possible to deform the decomposed patches (as shown by the deflation rule) without producing gaps or overlaps. The ansatz for matching rules is, as before, the set of the decorated vertex configurations, but, conversely to the former cases, this set consists

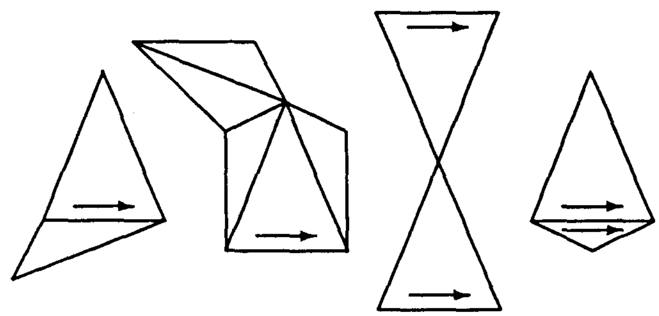

Fig. 5.1. Local decoration rules for eightfold symmetric triangular tiling $\mathcal{T}_{D_4}^{(8)}$. (One firstly decorates all acute triangles and the obtuse ones afterwards. In the third figure the decoration of one triangle is allways enforced by the first two figures.)

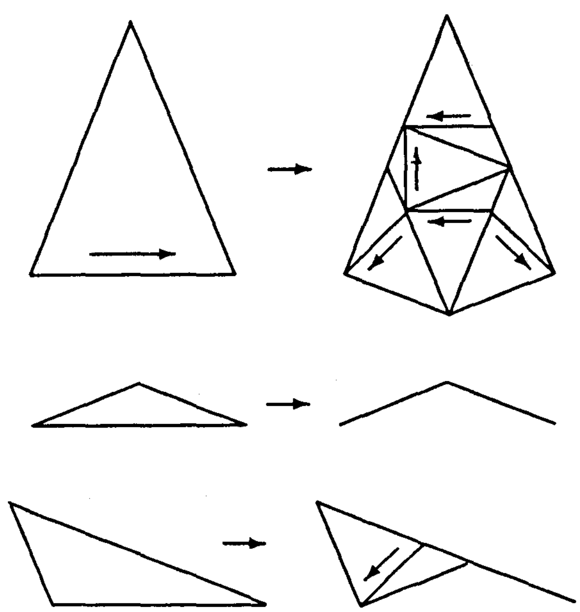

Fig. 5.2. Local deflation rules for $\mathcal{T}_{D_4}^{(8)}$.

now of the two distinct vertex classes mentioned above. This becomes important if one wants to construct the vertex decorations, because the two classes have to be treated separately. On the "Ammann-Beenker class", the original deflation method is also applicable. In either case, the vertex decorations of $\mathcal{T}_{AB}$, cf. p. 556f of Ref. 8, are reproduced. On the other class, only the generalized sector method (previous section) applies and yields a second kind of vertex decoration, compare Fig. 5.4.

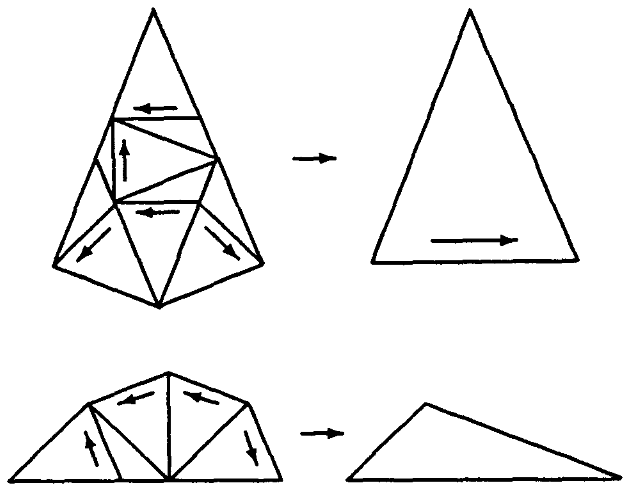

Fig. 5.3. Local inflation rules for $\mathcal{T}_{D_4}^{(8)}$. (This is sufficient: the holes are only obtuse triangles, whose decoration can be derived by means of Fig. 5.1).

That in the first case the decorations of $\mathcal{T}_{AB}$ are reproduced by the original method can be seen already after the first deflation. By means of this transformation, the sites of the original vertices of $\mathcal{T}_{AB}$ become exactly those vertices with eight acute triangles around. At the bases of these triangles, there will always be an oblique one. The outer shapes of these configurations are exactly those vertex decorations mentioned before. Moreover, right in between these configurations, there are pairs of oblique triangles situated at the edges of the tiling $\mathcal{T}_{AB}$. These pairs show exactly the right orientation for the edge decoration of $\mathcal{T}_{AB}$. Therefore, $\mathcal{T}_{D_4}^{(8)}$ implies both, the tiling $\mathcal{T}_{AB}$ *and* its decorations used for matching rules, in a local manner. Whereas, just as $\mathcal{T}_{NMG}$, $\mathcal{T}_{AB}$ without decoration is not in the same SMLD class as its decorated version. Now, we show that the SMLD class of the *decorated* tiling is the same as that of $\mathcal{T}_{D_4}^{(8)}$: The decorated tiles of $\mathcal{T}_{AB}$ imply a unique decomposition into the triangles of $\mathcal{T}_{D_4}^{(8)}$. Hence, the missing vertex class and the bonds of $\mathcal{T}_{D_4}^{(8)}$ can be reconstructed.

6. Concluding Remarks

Having demonstrated constructively that perfect matching rules for *undecorated* tilings do exist not only for the Penrose pattern, but also for triangular patterns with ten-, twelve-, and eightfold symmetry, the generality of these findings comes into question. First of all, perfect matching rules are relatively rare, although recent results on the generalized Penrose tilings[28] indicate that many important structures do have them. Let us therefore consider SMLD classes that are compatible with perfect matching rules. Then, we have two possible situations: either the decorations related to the matching rules are locally derivable from the SMLD class (and

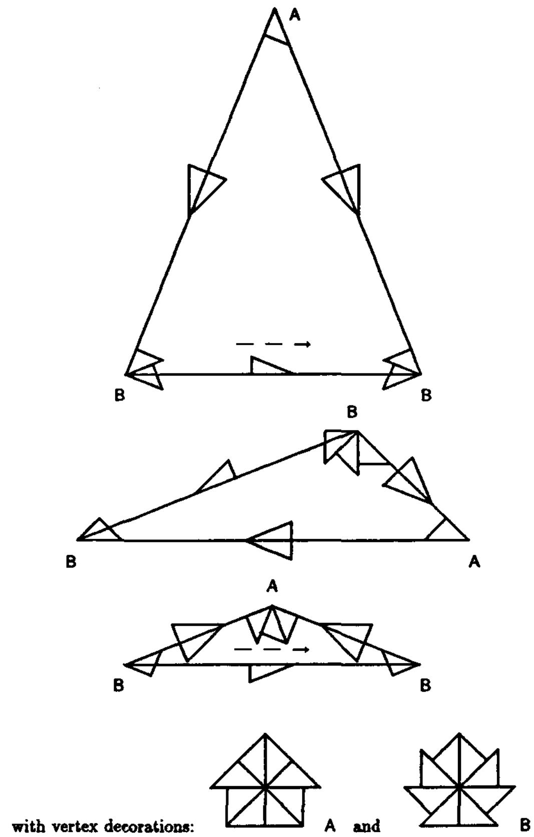

Fig. 5.4. Local matching rules for $\mathcal{T}_{D_4}^{(8)}$. Note that there are different edge decorations for different edge sizes. (The dashed arrows indicate the correspondence to the decoration rules of Fig. 5.1.)

therefore belong to it) or they are not (and therefore form another SMLD class). In the positive case, let us speak, for simplicity, of class immanent matching rules.

In view of our results, it looks possible that one can find tilings with class immanent matching rules and local inflation/deflation symmetry for many noncrystallographic symmetries. Examples are also known for icosahedrally symmetric tilings, e.g., for the Socolar-Steinhardt[12] and the Danzer tiling[13] which turn out to be both in the same SMLD class.[29,30]

Within the huge number of possible SMLD classes of a given point symmetry, it might be a relevant and interesting question to classify those which possess perfect matching rules.

Several methods are known to prove that given matching rules are perfect, but a systematic approach — eventually via the embedding into higher dimensions — is still to be developed. The treatment of the generalized Penrose tilings[28] might point in the right direction. With such a general method at hand, one could hope to come to a classification which we think would be an important step in the understanding of aperiodic structures.

Acknowledgments

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Appendix

In Sec. 4 it has been mentioned that the tiling $\mathcal{T}_{D_4}^{(12)}$ cannot be reconstructed from the sites of the vertices alone, i.e., the vertex set belongs to another SMLD class. This is due to the fact that the acceptance domains of the vertices (the three squares) decay into domains of tiles which have angles of $2\pi n/24$ with not necessarily even n. This corresponds in the physical space of the tiling to the occurrence of patches consisting of an acute triangle surrounded by two obtuse ones and a small regular one at its base. The outer shape of this patch is symmetric and it is possible to flip the arrangement.

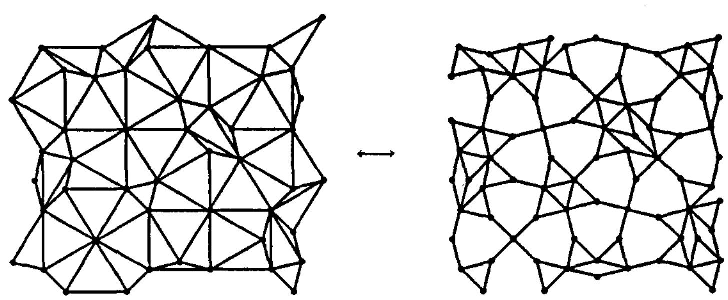

Fig. A.1. $\mathcal{T}_{D_4}^{(12)}$ and a tiling derived from it by introducing the circumcenters of the acute triangles. Both belong to the same SMLD class.

It is simple to get rid of this indefiniteness of edge or tile arrangement by introducing additional vertex points. From the patch mentioned it becomes suggestive to indicate the positions of the acute triangles, for example, by adding their circumcenters. Now, the new tiles become derivable from the vertices alone. Apart from that, only one edge length is needed (the smaller one of $\mathcal{T}_{D_4}^{(12)}$). The new tiles are the shield, blown up from the big regular triangle, the rhombus, blown up from the obtuse triangle, and the regular triangle, which are on one hand the originally small and on the other hand the reduced acute triangles.

The acceptance domains of the new vertices remain just the same as for $\mathcal{T}_{D_4}^{(12)}$, only a fourth one for the additional vertices is added. In Sec. 4 it has been mentioned that the circumcenters of the acute triangles are just the sites of the vertices of

$\mathcal{T}_{NMG}$. Therefore, the needed additional domain is the one of that tiling: a regular dodecagon. The circumradius of it has thereby the same length as the edges of the quadratic domains. Furthermore, it is clear that this is a tiling which belongs to the same SMLD class as $\mathcal{T}_{D_4}^{(12)}$. (For the backward direction all vertices with three incident edges have to be replaced by an acute triangle.) So we have, just as in Sec. 3, a twelvefold symmetric pattern which has perfect matching rules *and* is derivable from the vertex sites alone. A similar procedure is possible also for $\mathcal{T}_{D_4}^{(8)}$, yielding a tiling[21] which consists of the rhombi and squares of $\mathcal{T}_{AB}$.

References

1. D. Shechtman, I. Blech, D. Gratias, and J. W. Cahn, *Phys. Rev. Lett.* **53**, 1951 (1984).
2. T. Ishimasa, H.-U. Nissen and Y. Fukano, *Phys. Rev. Lett.* **55**, 511 (1984).
3. L. Bendersky, *Phys. Rev. Lett.* **55**, 1461 (1985).
4. N. Wang, H. Chen and K. H. Kuo, *Phys. Rev. Lett.* **59**, 1010 (1987).
5. R. S. Becker and A. R. Kortan, in *Quasicrystals — The State of the Art*, ed. D. P. DiVincenzo and P. J. Steinhardt (World Scientific, Singapore, 1991), Chap. 5.
6. H.-U. Nissen, this volume.
7. K. Ingersent, in *Quasicrystals — The State of the Art*, ed. D. P. DiVincenzo and P. J. Steinhardt (World Scientific, Singapore, 1991), Chap. 7.
8. B. Grünbaum and G. C. Shephard, *Tilings and Patterns* (W. H. Freeman., New York, 1987).
9. J. E. S. Socolar, *Phys. Rev.* **B39**, 10519 (1988).
10. F. Gähler, "Matching rules for quasicrystals: the composition-decomposition method", preprint, to appear in *J. Non-cryst. Solids.*
11. F. P. M. Beenker, Ph.D. Thesis 05B45, Eindhoven Univ. of Technology, The Netherlands, 1982. Remark: the matching rules (which are due to R. Ammann) are presented on p. 556f of Ref. 8.
12. J. E. S. Socolar and P. J. Steinhardt, *Phys. Rev.* **B34**, 617 (1986).
13. L. Danzer, *Discr. Math.* **76**, 1 (1989).
14. L. S. Levitov, *Commun. Math. Phys.* **119**, 627 (1988).
15. Z. Olami and S. Alexander, *Phys. Rev.* **B37**, 3973 (1988).
16. M. Baake, M. Schlottmann and P. D. Jarvis, *J. Phys.* **A24**, 4637 (1991).
17. R. Ammann's tiling(s), published on p. 550ff of Ref. 8 or recently in Ref. 18.
18. R. Ammann, B. Grünbaum and G. C. Shephard, *Discr. Comput. Geom.* **8**, 1 (1992).
19. M. Baake, D. Joseph and M. Schlottmann, *Int. J. Mod. Phys.* **B5**, 1927 (1991).
20. H.-U. Nissen, priv. comm. (1990).
21. M. Baake, R. Klitzing, H.-U. Nissen, and M. Schlottmann, in preparation.
22. R. Lück, *Proc. 7th Int. Conf. on Liquid and Amorphous Metals* (Kyoto, 1989).
23. M. Baake, P. Kramer, M. Schlottmann, and D. Zeidler, *Mod. Phys. Lett.* **B4**, 249 (1990) and *Int. J. Mod. Phys.* **B4**, 2217 (1990).
24. M. Baake, F. Gähler and M. Schlottmann, in preparation.
25. K. Niizeki and H. Mitani, *J. Phys.* **A20**, L405 (1987).
26. F. Gähler, in *Quasicrystalline Materials*, ed. Ch. Janot and J. M. Dubois (World Scientific, Singapore, 1988), p. 272.
27. L. Danzer, *Proc. 18th Int. Coll. Group Theor. Meth. Phys.*, Moscow, ed. V. V. Dodonov and V. I. Man'ko (Springer, Berlin, 1991).
28. A. Katz and L. S. Levitov, in preparation.
29. J. Roth, *J. Phys. A*, in press.
30. L. Danzer and A. Talis, this volume.